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Discrete Mathematics 306 (2006) 3074–3077

DISCRETE MATHEMATICS

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The number of equivalence classes of symmetric sign patterns

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Received 3 December 2003; received in revised form 29 September 2004; accepted 24 October 2004
Available online 14 July 2006

Abstract

This paper shows that the number of sign patterns of totally non-zero symmetric n-by-n matrices, up to conjugation by permutation and signature matrices and negation, is equal to the number of unlabelled graphs on n vertices.

Keywords: Symmetric matrix; Sign pattern; Enumeration; Duality

1. Introduction

By a (totally non-zero) sign pattern, we mean an n-by-n array $S=(s_{ij})$ of $+$ and $-$ signs. With S, we associate the set $\mathscr{S}$ of all real n-by-n matrices such that $A=(a_{ij})\in\mathscr{S}$ if and only if $a_{ij}>0$ whenever $s_{ij}=+$ and $a_{ij}<0$ whenever $s_{ij}=-$. Many questions about which properties P are *required* (all elements of $\mathscr{S}$ enjoy P) or *allowed* (some element of $\mathscr{S}$ has P) by a particular sign pattern S have been studied (see, for example, [4,5]).

One may also consider properties of pairs of sign patterns S_1 and S_2. For example, one such property of recent interest is commutativity of symmetric sign patterns [1]. We say that S_1 and S_2 *allow commutativity* ("commute", for short) if there exist symmetric matrices $A_1\in\mathscr{S}_1$ and $A_2\in\mathscr{S}_2$ such that $A_1A_2=A_2A_1$. For a given n, a complete answer to the above question might be succinctly described by an undirected graph G, whose vertices are the sign patterns, with an edge between two distinct sign patterns if and only if they commute. (Of course, any sign pattern commutes with itself.) We may think of the neighbourhood of S in G as the "commutant" of S.

It is clear that G may be completely described, using knowledge of the commutants of relatively few of the vertices of G. This is because of the following. We call an n-by-n matrix T a *signature matrix* if it is a diagonal matrix, whose diagonal entries are ± 1. If S is a sign pattern, then the signature similarity TST is an unambiguous sign pattern as well. It is elementary that two sign patterns S_1 and S_2 commute if and only if TS_1T and TS_2T commute, for any signature matrix T. Moreover, S_1 and S_2 commute if and only if $P^\top S_1P$ and $P^\top S_2P$ commute for any permutation matrix P, and also if and only if $-S_1$ and $-S_2$ commute. Thus, if the commutant of S is known, then the commutant of any matrix obtained from S by a combination of signature similarity, permutation similarity, and negation (in any order) is known as well. Empirical evidence suggests that there is no broader sequence of transformations which predictably transform the commutant of a given sign pattern.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.10.029

Thus, the equivalence classes of sign patterns under these transformations are of interest, both the nature of the classes and their number. Here, our interest is in the number of equivalence classes of totally non-zero symmetric sign patterns. Let $\mathscr{X}$ be the set of all such n-by-n sign patterns, and $f_1(n)$ the number of equivalence classes under the relation $\equiv$ generated by signature similarity, permutation similarity and negation. Moreover, let $f_2(n)$ be the (known) number of undirected graphs on n vertices up to isomorphism (that is, the unlabelled graphs). We show the following results:

Theorem 1. *With the above notation*, $f_1(n) = f_2(n)$.

Theorem 2. *If n is odd*, *there is a natural bijection between the set of equivalence classes of $\equiv$ on $\mathscr{X}$ and the set of unlabelled graphs on n vertices.*

Of course, for odd n, the first theorem follows from the second. But for even n, there is no such natural bijection.

2. Switching classes and even graphs

An *even graph* is one all of whose valencies are even. Such a graph, if connected, is *Eulerian*; that is, there is a path passing through each edge once and returning to its starting point. Unlabelled even graphs on n vertices were enumerated by Liskovec [6] and Robinson [8].

The operation of *switching* of a graph was defined by Seidel [9], and works as follows. Choose a subset X of vertices; replace edges between X and its complement by non-edges, and non-edges by edges, keeping adjacency within or outside X the same. Switching is an equivalence relation on the set of graphs on a given set of vertices. An *automorphism* of a switching class is a permutation of the vertices which preserves the class; it is enough that it maps one graph in the class to another.

While compiling information for the original version of the *Encyclopedia of Integer Sequences*, Neil Sloane found empirically that the numbers of unlabelled switching classes (isomorphism types of switching classes) and of unlabelled even graphs coincide. Mallows and Sloane [7] gave a proof of this fact, and another proof was given by Cameron [2].

We now outline both of these proofs, since we need to extend the ideas in each to prove Theorem 1.

The Mallows–Sloane proof. We use the Orbit-Counting Lemma (see, for example, [3, Theorem 2.2]). Unlabelled n-element structures of any type are defined to be orbits of the symmetric group S_n on labelled structures of that type (on the set $\{1, \ldots, n\}$). The number of orbits is just the average number of fixed points of the elements of S_n. So to show equality of the numbers of unlabelled structures, it is enough to show that any permutation fixes equally many structures of each type.

For even graphs, Liskovec and Robinson showed that the number of even graphs fixed by the permutation g is equal to $2^{c_2(g)-c(g)-\sigma(g)+1}$, where $c(g)$ is the number of cycles of the permutation g, $c_2(g)$ is the number of cycles in the action of g on the set of 2-element subsets of $\{1, \ldots, n\}$, and

$$\sigma(g) = \begin{cases} 1 & \text{if all cycles of } g \text{ have even length,} \\ 0 & \text{otherwise.} \end{cases}$$

Now the Orbit-Counting Lemma shows that the number of unlabelled even graphs is the average of this function over the symmetric group.

The key to the Mallows–Sloane proof is the following fact:

> If a permutation g fixes a switching class of graphs, then it fixes a graph in the switching class.

It then follows that the number of fixed graphs in the switching class is equal to the number of complementary pairs of subsets which are fixed by g, which is easily seen to be $2^{c(g)+\sigma(g)-1}$, with the same notation as before. Since the total number of fixed graphs of g is $2^{c_2(g)}$, we see that the number of fixed switching classes is indeed $2^{c_2(g)-c(g)-\sigma(g)+1}$, and the proof is complete. □

Cameron's proof. We convert the problem to one involving vector spaces over the field $\mathbb{F}$ with two elements. Let $V_2 = \mathbb{F}^{X^{\{2\}}}$, the set of functions from $X^{\{2\}}$ to $\mathbb{F}$, where $X = \{1, 2, \ldots, n\}$ and $X^{\{2\}}$ is the set of 2-element subsets of X.

The elements of V_2 are naturally identified with graphs on the vertex set X. Moreover, let V_1 be the set of functions from X to $\mathbb{F}$.

For any set Y, we use the natural inner product (a non-degenerate symmetric bilinear form, though not of course positive definite) on $\mathbb{F}^Y$ given by

$$f \cdot g = \sum_{y \in Y} f(y)g(y).$$

The characteristic functions of the sets of size 1 form an orthonormal basis for $\mathbb{F}^Y$ with respect to this inner product.

We have a map $\partial : V_2 \to V_1$ given by

$$(\partial h)(x) = \sum_{y \neq x} h(\{x, y\}).$$

The kernel of ∂ is the set of even graphs, since the above sum is zero in $\mathbb{F}$ if and only if the degree of x is even.

The dual to ∂ is the map $\delta : V_1 \to V_2$ given by

$$(\delta f)(\{x, y\}) = f(x) + f(y)$$

whose image is the set of complete bipartite graphs. (The graph δf has all edges between $f^{-1}(0)$ and $f^{-1}(1)$.) This space is spanned by the *stars* $K_{1,n-1}$ (which have the form δf, where f is the characteristic function of a singleton). Now a graph is even if and only if it is orthogonal to all stars. Hence $\mathrm{Ker}(\partial) = \mathrm{Im}(\delta)^{\perp}$.

The operation of switching with respect to a subset Y of X is realised by adding (in V_2) the complete bipartite graph with parts Y and $X \backslash Y$. So the quotient space $V_2/\mathrm{Im}(\delta)$ is the set of switching classes of graphs on X, which is thus naturally isomorphic to the dual space of $\mathrm{Ker}(\partial)$. This isomorphism respects the natural action of the symmetric group S_n. Since the number of orbits of a linear group on a finite vector space and its dual are equal (by Brauer's Lemma), the result is proved. $\square$

3. Proof of Theorem 1

We use the notation V_1 and V_2 as above, and let $V = V_1 \oplus V_2$. There is an obvious identification of $\mathscr{X}$ with V: the matrix A corresponds to (f, h), where $A_{ii} = (-1)^{f(i)}$ and $A_{ij} = A_{ji} = (-1)^{h(\{i,j\})}$ for $i \neq j$.

The transformations of signature similarity and negation defined in Section 1 correspond in V to the maps $(f, h) \mapsto (f, h + b)$, where b is a complete bipartite graph, and $(f, h) \mapsto (f + 1, h + 1)$ (where 1 denotes the constant function with value 1). So the equivalence classes of the relation generated by signature similarity and negation are the cosets of the subspace $W = \langle W', (1, 1)\rangle$, where W' consists of all $(0, b)$ for complete bipartite graphs b on X. Note that W is invariant under the symmetric group S_n, and $f_1(n)$ is the number of orbits of S_n on V/W. Also, $f_2(n)$ is the number of orbits of S_n on V_2.

By the Orbit-Counting Lemma, the theorem will follow if we can show:

> For any $g \in S_n$, the numbers of fixed points of g in V/W and in V_2 are equal.

For $g \in S_n$, let $c_1(g)$ and $c_2(g)$ be the numbers of cycles of g in its actions on X and $X^{(2)}$, respectively. Clearly g fixes $2^{c_2(g)}$ elements of V_2. Now an automorphism of a vector space has equally many fixed points on the space and its dual. The dual of V/W is naturally isomorphic to $W^{\perp}$. So we must show:

> For any $g \in S_n$, the number of fixed points of g in $W^{\perp}$ is equal to $2^{c_2(g)}$.

Now, as we saw earlier, a graph h is orthogonal to all complete bipartite graphs if and only if it is even. So, if W' consists of all $(0, h) \in V$ such that $h \in V_2$ is complete bipartite, then $(W')^{\perp}$ consists of all $(f, h) \in V$ such that $f \in V_1$ is arbitrary and $h \in V_2$ is an even graph. Moreover, (f, h) is orthogonal to $(1, 1)$ if and only if $|f| + |E(h)|$ is even.

As we saw in the last section, Liskovec and Robinson showed that, for $g \in S_n$, the number of even graphs fixed by g is $2^{c_2(g)-c(g)-\sigma(g)+1}$. The number of subsets of X fixed by g is $2^{c(g)}$. So the number of fixed points of g in $(W')^{\perp}$ is $2^{c_2(g)-\sigma(g)+1}$. Thus, we are finished if we can show the following:

> If $\sigma(g) = 1$, then every pair (f, h) (with h an even graph) fixed by g has $|f| + |E(h)|$ even; if $\sigma(g) = 0$, then exactly half of such pairs do.

Suppose first that $\sigma(g)=0$, so that g has a cycle of odd length. Then, of all the subsets of X fixed by g, half of them have even cardinality and half have odd cardinality; so half satisfy $|f|+|E(h)|$ even, for any h.

Next, suppose that $\sigma(g)=1$. Now all fixed sets of g in X have even cardinality, so we have to show that every even graph h fixed by g has an even number of edges.

The edge set of such an h is the union of some cycles of g on $X^{\{2\}}$. These cycles all have even length except for cycles consisting of opposite points in a cycle of g on X with length congruent to $2 \bmod 4$.

Let S be the set of cycles of g with length congruent to $2 \bmod 4$. Construct a graph G on the vertex set S as follows. The pair C, C' is joined by an edge if and only if the number of edges from a point in C to C' is odd. (This relation is symmetric.) The vertex C is coloured black if its opposite points are joined in h, and white otherwise. Since h is an even graph, the black vertices of G have odd degree while the white vertices have even degree. By the Handshaking Lemma, the number of black vertices is even. So an even number of odd cycles of g in $X^{\{2\}}$ consist of edges, and the total number of edges is even. Our final claim is established, and the theorem is proved.

4. Proof of Theorem 2

Suppose that n is odd. Call an element of $\mathscr{X}$ *special* if the product of the elements in each row is $+1$. To any special matrix A corresponds a unique graph on $\{1,\ldots,n\}$, in which i and j are adjacent if and only if $A_{ij}=-1$. Moreover, this correspondence is preserved by permutations. So we have to show that any equivalence class under signature similarity and negation contains a unique special matrix.

If S is the diagonal matrix with $S_{ii}=-1$ and $S_{jj}=+1$ for $j \neq i$, then the row products in SAS agree with those of A for the ith row and disagree in all others. The set of such matrices generates the group of signature matrices. So by signature similarities we can make all row products equal, viz. all $+1$ or all -1 depending on the parity of the number of negative row products in the original matrix. (Here we use the fact that n is odd.) Now by negation if necessary, we obtain a special matrix. The uniqueness is clear.

Remark. This proof can be reformulated in vector space language. It is easily checked that the vectors corresponding to special matrices form an S_n-invariant complement U to W, and projection from U to V_2 is an isomorphism.

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